

Separation of soft and collinear singularities from one-loop N -point integrals

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Abstract:

The soft and collinear singularities of general scalar and tensor one-loop N -point integrals are worked out explicitly. As a result a simple explicit formula is given that expresses the singular part in terms of 3-point integrals. Apart from predicting the singularities, this result can be used to transfer singular one-loop integrals from one regularization scheme to another or to subtract soft and collinear singularities from one-loop Feynman diagrams directly in momentum space.

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1 Introduction

Many interesting high-energy processes at future colliders, such as the LHC and an e^+e^- linear collider, lead to final states with more than two particles, rendering precise predictions much more complicated than for $2 \rightarrow 2$ particle reactions. The necessary calculation of radiative corrections bears additional complications and requires a further development of calculational techniques, as recently reviewed in Ref. [1].

Full one-loop calculations for processes with more than two final-state particles require, for instance, the evaluation of scalar and tensor N -point integrals. For $N = 5, 6$ several approaches have been proposed in the literature [2–5]. In this context the two main complications concern a numerically stable evaluation of tensor integrals on the one hand and a proper separation of infrared (soft and collinear) singularities on the other. In Ref. [5] the direct reduction of scalar 5-point to 4-point integrals, as proposed in Ref. [2], has been extended to tensor integrals. The nice feature in this approach is the avoidance of leading inverse Gram determinants, which necessarily appear in the well-known Passarino–Veltman reduction [6] and lead to numerical instabilities at the phase-space boundary.

In this paper we focus on the treatment of infrared (IR) or so-called “mass singularities” at one loop. According to Kinoshita [7], such mass singularities arise from two configurations that both lead to logarithmic singularities. *Collinear* singularities appear if a massless external particle splits into two massless internal particles of a loop diagram, and *soft* singularities arise if two external (on-shell) particles exchange a massless particle. If the involved particles are not precisely massless, the corresponding singularities show up as large logarithms involving the small masses, like $\ln(m_e/Q)$ where m_e is the electron mass and Q a large scale. If the masses involved in the singular configurations are exactly zero, the singularities appear as regularized divergences, such as $1/\epsilon$ poles where $\epsilon = (4 - D)/2$ in D -dimensional regularization. In both cases an analytical control over such terms is highly desirable, either in order to perform cancellations at the analytical level or to carry out resummations. In the following we show how to extract mass singularities from general tensor N -point integrals and finally give an explicit result for the singular parts in terms of related 3-point integrals. Moreover, we describe several ways how this result can be exploited and give some examples illustrating the easy use of our final result. The method to derive the general result of this paper was already applied in Ref. [8] to specific 5-point integrals, which appear in the next-to-leading order QCD corrections to the processes $gg/q\bar{q} \rightarrow t\bar{t}H$.

Mass singularities of the virtual and real radiative corrections are intrinsically connected in field theory. In fact, the famous Kinoshita–Lee–Nauenberg (KLN) theorem [7,9] states that the singularities completely cancel in sufficiently inclusive quantities. The result of this paper allows for a simple analytical handling of the mass singularities of the virtual one-loop corrections. The singular structure of the real corrections induced by one-parton emission (which corresponds to the one-loop level) can be easily read from so-called subtraction formalisms [10] which are designed for separating these singularities. Using these results and the KLN theorem, the one-loop singular structure of complete QCD and SUSY-QCD amplitudes has been derived in a closed form in Ref. [11]. Note that the attitude of this paper is rather different, since the one-loop integrals are inspected themselves, eventually leading to a prescription for extracting the singularities diagram by diagram.

The paper is organized as follows: In Section 2 we set our conventions and describe the situations in which mass singularities appear at one loop. Section 3 contains the actual separation of the mass singularities and our final result at the end of the section. In Section 4 we

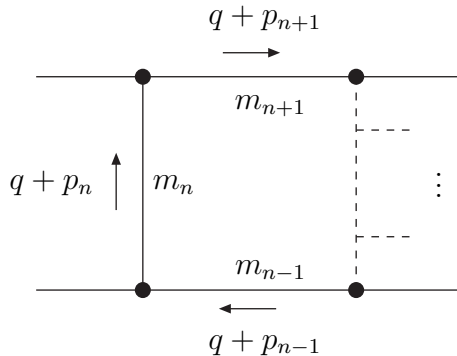


Figure 1: Momentum and mass assignment for a general one-loop N -point integral (2.1)

describe various ways to make use of the result and present some explicit applications. Section 5 contains a short summary. In the appendices we present a proof of an auxiliary identity used in Section 3 and provide a list of mass-singular scalar 3-point integrals that frequently appear in applications.

2 One-loop N -point integrals and mass singularities

We consider the general one-loop N -point integrals

$$T_{\mu_1 \dots \mu_P}^{(N)}(p_0, \dots, p_{N-1}, m_0, \dots, m_{N-1}) = \frac{(2\pi\mu)^{(4-D)}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{N_0 \cdots N_{N-1}}, \quad (2.1)$$

with the denominator factors

$$N_n = (q + p_n)^2 - m_n^2 + i0, \quad n = 0, \dots, N-1. \quad (2.2)$$

A diagrammatic illustration is shown in Figure 1. Note that we do not set the momentum p_0 to zero, as it is often done by convention, but keep this variable in order to facilitate a generic treatment of related integrals. In particular, with this convention all $T_{\dots}^{(N)}$ are invariant under exchange of any two propagator denominators N_n , or equivalently of two pairs of (p_n, m_n) . We follow the usual convention to denote N -point integrals with $N = 1, 2, \dots$ as

$$T^{(1)} \equiv A, \quad T^{(2)} \equiv B, \quad T^{(3)} \equiv C, \quad T^{(4)} \equiv D, \quad T^{(5)} \equiv E, \dots \quad (2.3)$$

Whenever the index n on momenta p_n or masses m_n exceeds the range $n = 0, \dots, N-1$, it is understood as modulo N , i.e. $n = 0$ and $n = N$ are equivalent, etc. For later use, we introduce the variables

$$c_{mn} = 2(p_{n+1} - p_n)(p_m - p_n), \quad d_{mn} = (p_m - p_n)^2 - m_m^2. \quad (2.4)$$

We are interested in “mass” singularities that appear if combinations of external squared momenta, $(p_{n+1} - p_n)^2$, and internal masses m_n become small, but do not consider singular configurations that are related to specific or isolated points in phase space, such as thresholds or forward scattering. We can, thus, distinguish two sets of parameters: one set that comprises all quantities $(p_m - p_n)^2$, m_n with fixed non-zero values, and another set of those quantities that are considered to be small, i.e. which formally tend to zero. In order to simplify the notation, we define an operation, indicated by a caret $\hat{}$ over a quantity X , which implies that all small quantities are set to zero in \hat{X} .

As shown by Kinoshita [7], “mass” (or “IR”) singularities can appear in one-loop diagrams in the following two situations:

1. An external line with a light-like momentum (e.g. a massless external on-shell particle) is attached to two massless propagators, i.e. there is an n with

$$(p_{n+1} - p_n)^2 \rightarrow 0, \quad m_{n+1} \rightarrow 0, \quad m_n \rightarrow 0 \quad \Rightarrow \quad \hat{d}_{n,n+1} = \hat{d}_{n+1,n} = 0. \quad (2.5)$$

The singularity is logarithmic and originates from integration momenta q with

$$q \rightarrow -p_n + x_n(p_n - p_{n+1}), \quad (2.6)$$

where x_n is an arbitrary real variable. Since the momentum $(q + p_n)$ on line n is then collinear to the external momentum $(p_n - p_{n+1})$, such singularities are called *collinear* singularities.

2. A massless particle is exchanged between two on-shell particles, i.e. there is an n with

$$\begin{aligned} m_n \rightarrow 0, \quad (p_{n-1} - p_n)^2 - m_{n-1}^2 \rightarrow 0, \quad (p_{n+1} - p_n)^2 - m_{n+1}^2 \rightarrow 0 \\ \Rightarrow \quad \hat{d}_{n-1,n} = \hat{d}_{n+1,n} = 0. \end{aligned} \quad (2.7)$$

The singularity is also logarithmic and originates from integration momenta q with

$$q \rightarrow -p_n, \quad (2.8)$$

i.e. the momentum transfer of the n th propagator tends to zero. Therefore, these singularities are called *soft* singularities.

In the following we focus on integrals with $N > 3$ and express the singular structure of one-loop N -point integrals in terms of 3-point integrals which are easily calculated with standard techniques, as for instance described in Refs. [4,6,12]. Of course, the same is true for the cases $N = 1, 2$, i.e. for tadpole and self-energy integrals, which are even simpler.

3 Separation of mass singularities

In this section, we first consider the asymptotic behaviour of the denominator of the integrand in Eq. (2.1) in the individual collinear and soft limits. Based on these partial results we derive a simple expression that resembles the whole integrand in all singular regions. Applying the loop integration to this expression directly leads to our main result which expresses the singular structure of an arbitrary N -point integral (2.1) with $N > 3$ in terms of 3-point integrals.

3.1 Asymptotic behaviour in collinear regions

For an integration momentum q in the collinear domain, as specified in Eq. (2.6), the two propagator denominators N_n and N_{n+1} tend to zero ($N_n, N_{n+1} \rightarrow 0$), and the N_k behave as

$$N_k \sim [-p_n + x_n(p_n - p_{n+1}) + p_k]^2 - m_k^2 = N_n + m_n^2 - x_n c_{kn} + d_{kn} \quad \text{for } k \neq n, n+1. \quad (3.1)$$

The collinear limit is mass singular if the external momentum squared $(p_{n+1} - p_n)^2$ and the two masses m_n, m_{n+1} are small. In this limit the two propagator denominators N_n, N_{n+1} tend to zero, but the others remain finite (for $x_n \neq 0, 1$):

$$N_k \rightarrow -x_n \hat{c}_{kn} + \hat{d}_{kn} \quad \text{for } k \neq n, n+1. \quad (3.2)$$

Note that the variable x_n is the only integration variable that is not fixed by the collinear limit. The product of all regular propagators N_k^{-1} can be decomposed into a sum over these propagators via taking the partial fraction,

$$\prod_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{1}{N_k} \sim \prod_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} = \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{A_{nk}^{\text{coll}}}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} = \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{A_{nk}^{\text{coll}}}{N_k}. \quad (3.3)$$

The coefficients A_{nk}^{coll} are functions of the variables \hat{d}_{kn} and \hat{c}_{kn} alone and thus fixed by the external kinematics. The explicit result for A_{nk}^{coll} reads

$$A_{nk}^{\text{coll}} = \frac{\hat{c}_{kn}^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^{N-1} (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})}, \quad (3.4)$$

as proven in App. A. The collinear singularity arises from the region where the propagator denominators N_n and N_{n+1} both become small with no preference to any of the two. In order to reveal this equivalence, we rewrite A_{nk}^{coll} using

$$\begin{aligned} c_{kn} &= 2(p_{n+1} - p_n)(p_k - p_n) \\ &= [(p_k - p_n)^2 - m_k^2] - [(p_k - p_{n+1})^2 - m_k^2] + (p_{n+1} - p_n)^2 \\ \Rightarrow \quad \hat{c}_{kn} &= \hat{d}_{kn} - \hat{d}_{k, n+1} \quad \text{for } (p_{n+1} - p_n)^2 \rightarrow 0 \end{aligned} \quad (3.5)$$

and

$$\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn} = \hat{d}_{kn} \hat{d}_{l, n+1} - \hat{d}_{ln} \hat{d}_{k, n+1}. \quad (3.6)$$

Inserting these relations, we get

$$A_{nk}^{\text{coll}} = \frac{(\hat{d}_{kn} - \hat{d}_{k, n+1})^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^{N-1} (\hat{d}_{kn} \hat{d}_{l, n+1} - \hat{d}_{ln} \hat{d}_{k, n+1})}, \quad (3.7)$$

in which the equivalence of the n th and $(n+1)$ th propagators is evident.

Multiplying Eq. (3.3) with $N_n^{-1} N_{n+1}^{-1}$ yields a relation, which is valid in the collinear limit, between the product of all N propagators and a linear combination of products $N_n^{-1} N_{n+1}^{-1} N_k^{-1}$ involving only three propagators,

$$\prod_{k=0}^{N-1} \frac{1}{N_k} \sim \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{A_{nk}^{\text{coll}}}{N_n N_{n+1} N_k}. \quad (3.8)$$

Thus, the collinear singularity associated with the propagators N_n^{-1} , N_{n+1}^{-1} in an N -point integral is expressed in terms of a sum of 3-point integrals involving the n th, the $(n+1)$ th, and any other line of the diagram.

3.2 Asymptotic behaviour in soft regions

The soft singularity connected with a massless propagator N_n arises from momenta $q \rightarrow -p_n$, where $N_{n-1}, N_n, N_{n+1} \rightarrow 0$. The other denominators tend to a regular limit in this case,

$$N_k \rightarrow (p_k - p_n)^2 - m_k^2 = \hat{d}_{kn} \quad \text{for } k \neq n-1, n, n+1, \quad (3.9)$$

and the product of all propagators behaves like

$$\prod_{k=0}^{N-1} \frac{1}{N_k} \sim \frac{A_n^{\text{soft}}}{N_{n-1} N_n N_{n+1}} \quad \text{with} \quad A_n^{\text{soft}} = \prod_{\substack{l=0 \\ l \neq n-1, n, n+1}}^{N-1} \frac{1}{\hat{d}_{ln}}. \quad (3.10)$$

We still have to consider the possibility that one or both ends of the soft line n is part of a collinear configuration treated above. If this is the case, the soft limit can be reached as limiting case of a collinear limit. Assuming n again as the soft line, the two “degenerate” collinear limits are $x_n \rightarrow 0$ and $x_{n-1} \rightarrow 1$. Both lead to $q \rightarrow -p_n$, but in the former case lines n and $n+1$ correspond to a collinear configuration, in the latter lines $n-1$ and n . It is quite easy to see that the soft asymptotic behaviour (3.10) is already correctly included in the collinear behaviour (3.8) in either case, because

$$A_{n,n-1}^{\text{coll}} = A_{n-1,n+1}^{\text{coll}} = A_n^{\text{soft}} \quad \text{for } \hat{d}_{n-1,n} = \hat{d}_{n+1,n} = 0. \quad (3.11)$$

3.3 Final result

From the above considerations it is clear that we obtain an expression for the asymptotic behaviour of the product of all propagator denominators in all collinear and soft regions upon adding the asymptotic expressions of all collinear and soft regions, which can be read from Eqs. (3.8) and (3.10), and carefully avoiding double-counting of soft asymptotic terms. To this end, we define

$$A_{nk} = \begin{cases} A_{nk}^{\text{coll}} & \text{if Eq. (2.5) is fulfilled, but} \\ & \text{neither } [k = n-1 \text{ and } \hat{d}_{n-1,n} = 0] \\ & \text{nor } [k = n+2 \text{ and } \hat{d}_{n+2,n+1} = 0], \\ A_n^{\text{soft}} & \text{if } k = n-1 \text{ and Eq. (2.7) is fulfilled,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.12)$$

With this definition we can write down the asymptotic behaviour valid for all collinear and soft regions as

$$\prod_{k=0}^{N-1} \frac{1}{N_k} \sim \sum_{n=0}^{N-1} \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{A_{nk}}{N_n N_{n+1} N_k}. \quad (3.13)$$

Obviously each soft part is included by the A_n^{soft} terms exactly once, and the collinear contributions from $A_{n,n-1}^{\text{coll}}$ and $A_{n-1,n+1}^{\text{coll}}$ are omitted if they are already covered by the soft terms A_n^{soft} . Integrating Eq. (3.13) over $\frac{(2\pi\mu)^{(4-D)}}{i\pi^2} \int d^D q$ on the l.h.s. yields the scalar integral $T_0^{(N)}$ and on the r.h.s. a linear combination of scalar 3-point integrals C_0 , which has exactly the same structure of collinear and soft singularities as $T_0^{(N)}$. An analogous relation is obtained for tensor integrals if the additional factor $q_{\mu_1} \cdots q_{\mu_P}$ is included in the integration, since this factor does not lead to additional singularities. Note that the asymptotic relation (3.13), which

describes the leading behaviour, is in fact sufficient to extract all mass singularities from the one-loop integral, since the degree of the singularities is logarithmic. In summary the complete mass-singular part of a general one-loop tensor N -point function reads

$$\begin{aligned}
& T_{\mu_1 \dots \mu_P}^{(N)}(p_0, \dots, p_{N-1}, m_0, \dots, m_{N-1}) \Big|_{\text{sing}} \\
&= \sum_{n=0}^{N-1} \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} A_{nk} C_{\mu_1 \dots \mu_P}(p_n, p_{n+1}, p_k, m_n, m_{n+1}, m_k).
\end{aligned} \tag{3.14}$$

The sum over n and k runs over all subdiagrams whose scalar integral develops a collinear or soft singularity. A tensor integral is, however, not necessarily mass singular if the related scalar integral develops such a singularity. For such tensor integrals the regular 3-point integrals on the r.h.s. of Eq. (3.14) could be dropped. We note that for $P \geq 2$, artificial ultraviolet singularities appear in the tensor 3-point integrals on the r.h.s. of Eq. (3.14). These can be regularized in dimensional regularization and easily separated from the mass singularities (see, e.g., the appendix of Ref. [5]).

In order to render the above result more useful, we present a list of mass-singular C_0 functions in App. B. This paper, thus, contains the needed ingredients to predict the mass singularities of most scalar N -point functions occurring in practice. To obtain the singularities of tensor integrals, only the 3-point tensor integrals have to be derived, which can be easily inferred with the well-known Passarino–Veltman algorithm [6] (see also Refs. [4,5]).

4 Discussion and applications

4.1 Possible applications of the final result

The relation (3.14) can be exploited in various directions:

- As pointed out in the previous section, the mass singularities of arbitrary N -point integrals can be easily derived from 3-point functions. This statement is true in any regularization scheme, i.e. for any N -point integral all small-mass logarithms and/or poles in $(D-4)$ in dimensional regularization can be easily inferred.
- The singular integral $T_{\mu_1 \dots \mu_P}^{(N)}|_{\text{sing}}$ can be used to translate any IR-divergent N -point integral from one regularization scheme to another. To this end, the regularization-scheme-independent difference

$$T_{\mu_1 \dots \mu_P}^{(N)} - T_{\mu_1 \dots \mu_P}^{(N)}|_{\text{sing}} = (\text{IR finite}) \tag{4.1}$$

is considered. For the translation from one scheme to the other only the singular part $T_{\mu_1 \dots \mu_P}^{(N)}|_{\text{sing}}$, and thus the relevant 3-point integrals, have to be known in the two regularization schemes.

- The trick described in the last item has been used in Ref. [8] to translate D -dimensional 5-point integrals into a mass regularization with $D=4$, in order to make use of the direct reduction [2,5] of 5-point to 4-point integrals, which works in four space-time dimensions. In this context it was observed that the formal relation between 5-point and 4-point integrals, which was derived in four dimensions, is also valid in D dimensions up to $\mathcal{O}(D-4)$ terms, since the extraction of the singularities works in any regularization

scheme with the same linear combination of 3-point integrals. From the results of this paper we conclude that this statement generalizes to arbitrary N -point integrals, i.e. the reduction of an N -point integral to 4-point integrals works in D dimensions in precisely the same way as in four dimensions [up to terms of $\mathcal{O}(D-4)$], without the appearance of extra terms.

- Since Eq. (3.14) has been derived in momentum space, it could also be used as local counterterm in the momentum-space integral, i.e. taking the difference in Eq. (4.1) before the integration over the loop momentum q , the integral becomes IR (soft and collinear) finite and can be evaluated without IR regulator. The loop integration of the subtracted part is extremely simple, because it involves only 3-point functions, and can be added again after the integration of the difference. This procedure could be very useful in purely numerical approaches to loop integrals, as e.g. described in Ref. [13].

In other words, Eq. (3.14) can serve as the basis of a subtraction formalism for one-loop corrections, very similar to the frequently used subtraction formalisms for real corrections as worked out in Ref. [10].

4.2 Sudakov limit of the one-loop box integral

As a simple application, we consider the box integral $D_{\dots}(p_0, p_1, p_2, p_3, m_0, m_1, m_2, m_3)$ in the so-called Sudakov limit, where all external squared momenta and internal masses are considered to be much smaller than the two Mandelstam variables

$$s = (p_2 - p_0)^2, \quad t = (p_3 - p_1)^2. \quad (4.2)$$

In this limit, there are four regions for soft singularities, and Eq. (3.14) yields

$$\begin{aligned} & D_{\dots}(p_0, p_1, p_2, p_3, m_0, m_1, m_2, m_3) \Big|_{\text{sing}} \\ &= \frac{1}{s} C_{\dots}(p_1, p_2, p_3, m_1, m_2, m_3) + \frac{1}{t} C_{\dots}(p_2, p_3, p_0, m_2, m_3, m_0) \\ &+ \frac{1}{s} C_{\dots}(p_3, p_0, p_1, m_3, m_0, m_1) + \frac{1}{t} C_{\dots}(p_0, p_1, p_2, m_0, m_1, m_2). \end{aligned} \quad (4.3)$$

For the scalar integral this is in agreement with Eq. (57) of Ref. [14], where the remaining finite contribution was derived as well. In Ref. [14] also tensor integrals up to rank 4 have been considered; the singularities predicted by Eq. (4.3) have been checked against these results. The soft singularities in the C_{\dots} functions on the r.h.s. of Eq. (4.3) arise from integration momenta $q \rightarrow p_2, p_3, p_0, p_1$, respectively. The singular terms in the tensor coefficients of the C_{\dots} functions can be related to the respective scalar C_0 functions rather easily. For instance, shifting the integration momentum $q \rightarrow q - p_2$ in the first C_{\dots} function on the r.h.s. of Eq. (4.3), power-counting in the shifted momentum q shows that terms with q in the numerator are not mass singular. Thus, in the first tensor C_{\dots} function only covariants built of the momentum p_2 alone receive singular coefficients that are all proportional to the respective C_0 function. The same reasoning applies to the other three C_{\dots} functions. In summary, the mass-singular terms of the tensor 4-point functions in the Sudakov limit are given by

$$\begin{aligned} & D^{\mu_1 \dots \mu_P}(p_0, p_1, p_2, p_3, m_0, m_1, m_2, m_3) \Big|_{\text{sing}} \\ & \sim (-1)^P \frac{p_2^{\mu_1} \dots p_2^{\mu_P}}{s} C_0(p_1, p_2, p_3, m_1, m_2, m_3) + (-1)^P \frac{p_3^{\mu_1} \dots p_3^{\mu_P}}{t} C_0(p_2, p_3, p_0, m_2, m_3, m_0) \end{aligned}$$

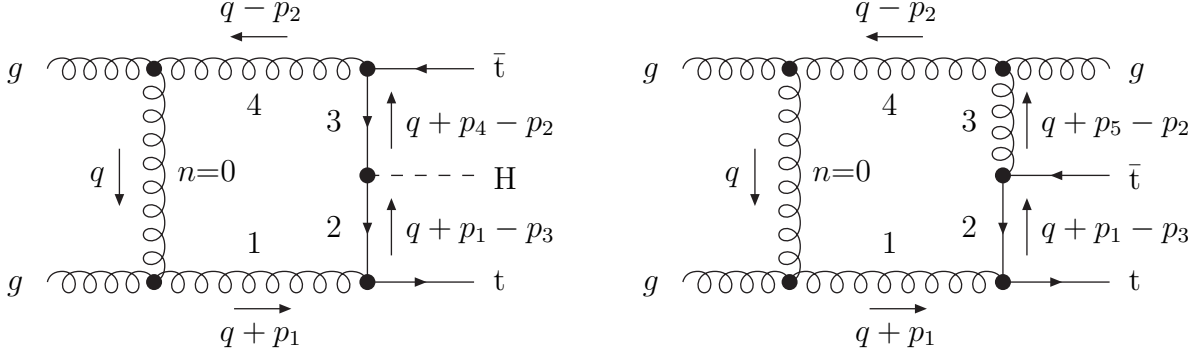


Figure 2: Examples for pentagon diagrams contributing to the one-loop corrections to the processes $gg \rightarrow t\bar{t}H$ and $gg \rightarrow t\bar{t}g$

$$+ (-1)^P \frac{p_0^{\mu_1} \cdots p_0^{\mu_P}}{s} C_0(p_3, p_0, p_1, m_3, m_0, m_1) + (-1)^P \frac{p_1^{\mu_1} \cdots p_1^{\mu_P}}{t} C_0(p_0, p_1, p_2, m_0, m_1, m_2), \quad (4.4)$$

where the \sim sign indicates that regular terms have been dropped, i.e. Eq. (3.14) is not applied literally.

4.3 Singular structure of some 5-point integrals

(i) *A 5-point integral for the process $gg \rightarrow t\bar{t}H$*

In Ref. [8] the three different types of IR-singular 5-point integrals that appear in the next-to-leading order (NLO) QCD correction to $gg/q\bar{q} \rightarrow t\bar{t}H$ have been calculated in dimensional regularization. One of the corresponding pentagon diagrams is shown on the l.h.s. of Figure 2. In order to make use of the direct reduction [2,5] of 5-point to 4-point integrals in four space-time dimensions, the dimensionally regulated integrals have been translated into a mass regularization by using the trick described in the previous section. However, the construction of the singular parts of the 5-point in terms of 3-point integrals has been done integral by integral. For the 5-point integral on the l.h.s. of Figure 2, which is a tensor integral of rank 4, we can easily verify that the explicit formula (3.14) yields the same result as quoted in Ref. [8]. Assigning the momenta according to

$$g(p_1) + g(p_2) \rightarrow t(p_3) + \bar{t}(p_4) + H(p_5) \quad (4.5)$$

and the defining

$$s = (p_1 + p_2)^2, \quad s_{ij} = (p_i + p_j)^2, \quad t_{kj} = (p_k - p_j)^2, \quad k = 1, 2, \quad i, j = 3, 4, 5, \quad (4.6)$$

the auxiliary quantities \hat{d}_{mn} read

$$(\hat{d}_{mn}) = \begin{pmatrix} 0 & 0 & t_{13} & t_{24} & 0 \\ 0 & 0 & m_t^2 & s_{35} & s \\ t_{13} - m_t^2 & 0 & -m_t^2 & M_H^2 - m_t^2 & s_{45} - m_t^2 \\ t_{24} - m_t^2 & s_{35} - m_t^2 & M_H^2 - m_t^2 & -m_t^2 & 0 \\ 0 & s & s_{45} & m_t^2 & 0 \end{pmatrix}, \quad (4.7)$$

where an obvious matrix notation is used. Inserting this into Eq. (3.14) and identifying the singular 3-point integrals yields

$$\begin{aligned}
& E_{\dots}(0, p_1, p_1 - p_3, p_4 - p_2, -p_2, 0, 0, m_t, m_t, 0) \Big|_{\text{sing}} \\
&= \frac{1}{s(s_{35} - m_t^2)} C_{\dots}(0, p_1, p_1 - p_3, 0, 0, m_t) \\
&\quad - \frac{(t_{24} - s_{35})^2}{s(s_{35} - m_t^2)(t_{13} - m_t^2)(t_{24} - m_t^2)} C_{\dots}(0, p_1, p_4 - p_2, 0, 0, m_t) \\
&\quad + \frac{1}{s(s_{45} - m_t^2)} C_{\dots}(0, p_4 - p_2, -p_2, 0, m_t, 0) \\
&\quad + \frac{1}{(t_{13} - m_t^2)(t_{24} - m_t^2)} C_{\dots}(0, p_1, -p_2, 0, 0, 0) \\
&\quad - \frac{(t_{13} - s_{45})^2}{s(s_{45} - m_t^2)(t_{13} - m_t^2)(t_{24} - m_t^2)} C_{\dots}(0, p_1 - p_3, -p_2, 0, m_t, 0), \tag{4.8}
\end{aligned}$$

in agreement with Eq.(2.37) of Ref. [8].

(ii) *A 5-point integral for the process $gg \rightarrow t\bar{t}g$*

As another example, we consider the 5-point integral corresponding to the diagram on the r.h.s. of Figure 2 which contributes to the (yet unknown) NLO QCD correction to $gg \rightarrow t\bar{t}g$. Analogously to the previous case, we assign the momenta according to

$$g(p_1) + g(p_2) \rightarrow t(p_3) + \bar{t}(p_4) + g(p_5) \tag{4.9}$$

and keep the definitions (4.6). The auxiliary quantities \hat{d}_{mn} read

$$(\hat{d}_{mn}) = \begin{pmatrix} 0 & 0 & t_{13} & t_{25} & 0 \\ 0 & 0 & m_t^2 & s_{34} & s \\ t_{13} - m_t^2 & 0 & -m_t^2 & 0 & s_{45} - m_t^2 \\ t_{25} & s_{34} & m_t^2 & 0 & 0 \\ 0 & s & s_{45} & 0 & 0 \end{pmatrix}. \tag{4.10}$$

Seven soft or collinear singular 3-point subdiagrams can be identified, and Eq. (3.14) yields

$$\begin{aligned}
& E_{\dots}(0, p_1, p_1 - p_3, p_5 - p_2, -p_2, 0, 0, m_t, 0, 0) \Big|_{\text{sing}} \\
&= \frac{1}{ss_{34}} C_{\dots}(0, p_1, p_1 - p_3, 0, 0, m_t) \\
&\quad - \frac{(t_{25} - s_{34})^2}{ss_{34}t_{25}(t_{13} - m_t^2)} C_{\dots}(0, p_1, p_5 - p_2, 0, 0, 0) \\
&\quad + \frac{1}{t_{25}s_{34}} C_{\dots}(p_1 - p_3, p_5 - p_2, -p_2, m_t, 0, 0) \\
&\quad + \frac{1}{s(s_{45} - m_t^2)} C_{\dots}(0, p_5 - p_2, -p_2, 0, 0, 0) \\
&\quad - \frac{(s - s_{34})^2}{ss_{34}t_{25}(s_{45} - m_t^2)} C_{\dots}(p_1, p_5 - p_2, -p_2, 0, 0, 0) \\
&\quad + \frac{1}{t_{25}(t_{13} - m_t^2)} C_{\dots}(0, p_1, -p_2, 0, 0, 0) \\
&\quad - \frac{(t_{13} - s_{45})^2}{st_{25}(s_{45} - m_t^2)(t_{13} - m_t^2)} C_{\dots}(0, p_1 - p_3, -p_2, 0, m_t, 0). \tag{4.11}
\end{aligned}$$

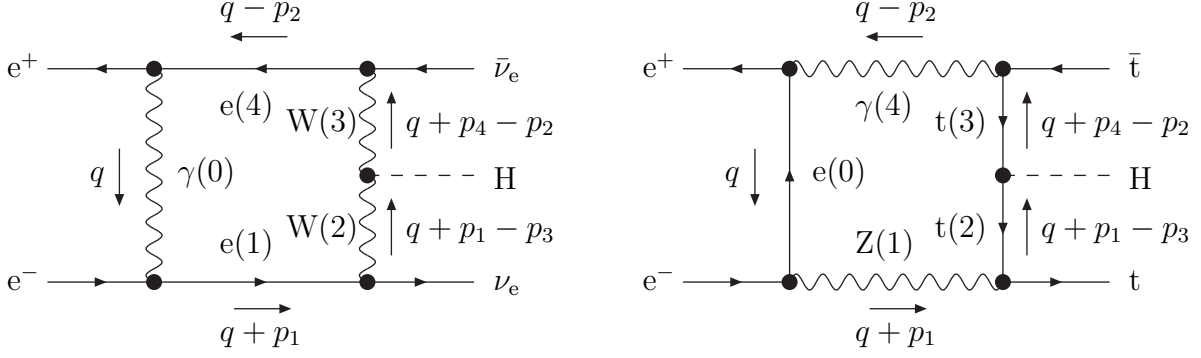


Figure 3: Examples for pentagon diagrams contributing to the one-loop corrections to the processes $e^+e^- \rightarrow \nu_e \bar{\nu}_e H$ and $e^+e^- \rightarrow t \bar{t} H$ (The number n of propagator N_n is indicated in parentheses.)

This result can again be used to relate the dimensionally regulated integral into any 4-dimensional regularization which then can be reduced to 4-point integrals using the method of Refs. [2,5].

(iii) A 5-point integral for the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e H$

Now we consider the diagram on the l.h.s. of Figure 3, which contributes to the $\mathcal{O}(\alpha)$ corrections of the process

$$e^-(p_1) + e^+(p_2) \rightarrow \nu_e(p_3) + \bar{\nu}_e(p_4) + H(p_5). \quad (4.12)$$

The soft singularity arising from photon exchange is regularized by the infinitesimally small photon mass λ . The electron mass m_e is considered to be much smaller than all other masses and scalar products, thus leading to mass-singular $\ln m_e$ terms. Again we make use of definition (4.6) for the kinematical variables. The auxiliary quantities \hat{d}_{mn} read

$$(\hat{d}_{mn}) = \begin{pmatrix} 0 & 0 & t_{13} & t_{24} & 0 \\ 0 & 0 & 0 & s_{35} & s \\ t_{13} - M_W^2 & -M_W^2 & -M_W^2 & M_H^2 - M_W^2 & s_{45} - M_W^2 \\ t_{24} - M_W^2 & s_{35} - M_W^2 & M_H^2 - M_W^2 & -M_W^2 & -M_W^2 \\ 0 & s & s_{45} & 0 & 0 \end{pmatrix}. \quad (4.13)$$

Inserting this into Eq. (3.14) and identifying the singular 3-point integrals yields

$$\begin{aligned} & E_{\dots}(0, p_1, p_1 - p_3, p_4 - p_2, -p_2, \lambda, m_e, M_W, M_W, m_e) \Big|_{\text{sing}} \\ &= \frac{t_{13}^2}{[(s_{35}t_{13} + M_W^2(t_{24} - t_{13} - s_{35}))s(t_{13} - M_W^2)]} C_{\dots}(0, p_1, p_1 - p_3, 0, m_e, M_W) \\ & \quad - \frac{(t_{24} - s_{35})^2}{[(s_{35}t_{13} + M_W^2(t_{24} - t_{13} - s_{35}))s(t_{24} - M_W^2)]} C_{\dots}(0, p_1, p_4 - p_2, 0, m_e, M_W) \\ & \quad + \frac{t_{24}^2}{[s_{45}t_{24} + M_W^2(t_{13} - t_{24} - s_{45})]s(t_{24} - M_W^2)} C_{\dots}(0, p_4 - p_2, -p_2, 0, M_W, m_e) \\ & \quad + \frac{1}{(t_{13} - M_W^2)(t_{24} - M_W^2)} C_{\dots}(0, p_1, -p_2, \lambda, m_e, m_e) \end{aligned}$$

$$- \frac{(t_{13} - s_{45})^2}{[s_{45}t_{24} + M_W^2(t_{13} - t_{24} - s_{45})]s(t_{13} - M_W^2)} C_{\dots}(0, p_1 - p_3, -p_2, 0, M_W, m_e), \quad (4.14)$$

where λ is set to zero in all C_{\dots} functions that are not soft singular. In Ref. [15], where the $\mathcal{O}(\alpha)$ corrections to the processes $e^+e^- \rightarrow \nu\bar{\nu}H$ have been worked out, the 5-point integral has been calculated directly from the corresponding 4-point integrals using the method of Refs. [2,5]. We have numerically verified the correctness of Eq. (4.14) by checking the difference between E_0 taken from Ref. [15] and the r.h.s. of Eq. (4.14) to be independent of $\ln \lambda$ and $\ln m_e$.

(iv) *A 5-point integral for the process $e^+e^- \rightarrow t\bar{t}H$*

Finally, we turn to the diagram on the r.h.s. of Figure 3, which contributes to

$$e^-(p_1) + e^+(p_2) \rightarrow t(p_3) + \bar{t}(p_4) + H(p_5) \quad (4.15)$$

at one loop. The masses λ and m_e are treated as in the previous example, and definition (4.6) is again employed. The auxiliary quantities \hat{d}_{mn} read

$$(\hat{d}_{mn}) = \begin{pmatrix} 0 & 0 & t_{13} & t_{24} & 0 \\ -M_Z^2 & -M_Z^2 & m_t^2 - M_Z^2 & s_{35} - M_Z^2 & s - M_Z^2 \\ t_{13} - m_t^2 & 0 & -m_t^2 & M_H^2 - m_t^2 & s_{45} - m_t^2 \\ t_{24} - m_t^2 & s_{35} - m_t^2 & M_H^2 - m_t^2 & -m_t^2 & 0 \\ 0 & s & s_{45} & m_t^2 & 0 \end{pmatrix}. \quad (4.16)$$

Equation (3.14) yields

$$\begin{aligned} & E_{\dots}(0, p_1, p_1 - p_3, p_4 - p_2, -p_2, m_e, M_Z, m_t, \lambda) \Big|_{\text{sing}} \\ &= \frac{1}{(s - M_Z^2)(s_{45} - m_t^2)} C_{\dots}(0, p_4 - p_2, -p_2, m_e, m_t, \lambda) \\ &+ \frac{s^2}{[s(t_{13} - m_t^2) + M_Z^2(s_{45} - t_{13})](s - M_Z^2)(t_{24} - m_t^2)} C_{\dots}(0, p_1, -p_2, m_e, M_Z, 0) \\ &- \frac{(t_{13} - s_{45})^2}{[s(t_{13} - m_t^2) + M_Z^2(s_{45} - t_{13})](s_{45} - m_t^2)(t_{24} - m_t^2)} \\ &\times C_{\dots}(0, p_1 - p_3, -p_2, m_e, m_t, 0). \end{aligned} \quad (4.17)$$

This relation has been numerically checked against the results in the calculation [16] of the $\mathcal{O}(\alpha)$ corrections to the process $e^+e^- \rightarrow t\bar{t}H$, as described in the previous example.

5 Summary

By analyzing the soft and collinear limits in one-loop N -point integrals, the mass singularities of general tensor one-loop N -point integrals are explicitly expressed in terms of 3-point integrals. The remarkably simple result is illustrated in some non-trivial examples, such as 5-point integrals occurring in $2 \rightarrow 3$ particle reactions.

The final formula holds true in any regularization scheme and, thus, can not only be used to predict the soft and collinear singularities of integrals, but also to translate mass-singular one-loop integrals from one regularization scheme to another. This, in particular, shows that

the reduction of N -point integrals with $N \geq 5$ to 4-point integrals in D dimensions proceeds as in four dimensions without extra contributions.

Since the separation of soft and collinear singularities is carried out in momentum space, the method can also be used as subtraction formalism for one-loop integrals, similar to the subtraction approaches used in the calculation of real radiative corrections.

Appendix

A Proof of an auxiliary identity

Here we make up for the proof of Eq. (3.4). We first proof the auxiliary identity

$$\sum_{k=0}^N \frac{b_k^{N-1}}{\prod_{\substack{l=0 \\ l \neq k}}^N (b_k a_l - b_l a_k)} = 0, \quad (\text{A.1})$$

where a_k, b_k ($k = 0, 1, \dots$) represent any two series of variables so that the denominators in the above expressions are non-zero. We proceed by induction. The case $N = 1$ is trivial, since then the sum consists of two terms with opposite sign. Now we assume Eq. (A.1) to be valid and deduce the analogous relation for $N \rightarrow N + 1$, i.e. we have to show that the following expression vanishes,

$$\begin{aligned} \sum_{k=0}^{N+1} \frac{b_k^N}{\prod_{\substack{l=0 \\ l \neq k}}^{N+1} (b_k a_l - b_l a_k)} &= \sum_{k=0}^{N-1} \frac{b_k}{(b_k a_N - b_N a_k)(b_k a_{N+1} - b_{N+1} a_k)} \frac{b_k^{N-1}}{\prod_{\substack{l=0 \\ l \neq k}}^{N-1} (b_k a_l - b_l a_k)} \\ &+ \frac{b_N^N}{(b_N a_{N+1} - b_{N+1} a_N) \prod_{l=0}^{N-1} (b_N a_l - b_l a_k)} \\ &+ \frac{b_{N+1}^N}{(b_{N+1} a_N - b_N a_{N+1}) \prod_{l=0}^{N-1} (b_{N+1} a_l - b_l a_k)}. \end{aligned} \quad (\text{A.2})$$

In the last equation we have only separated the last two terms from the sum and extracted some factors for convenience. Now we rewrite a factor of the first term on the r.h.s. as follows,

$$\begin{aligned} &\frac{b_k}{(b_k a_N - b_N a_k)(b_k a_{N+1} - b_{N+1} a_k)} \\ &= \frac{1}{b_N a_{N+1} - b_{N+1} a_N} \left(\frac{b_N}{b_k a_N - b_N a_k} - \frac{b_{N+1}}{b_k a_{N+1} - b_{N+1} a_k} \right), \end{aligned} \quad (\text{A.3})$$

and obtain

$$\sum_{k=0}^{N+1} \frac{b_k^N}{\prod_{\substack{l=0 \\ l \neq k}}^{N+1} (b_k a_l - b_l a_k)}$$

$$\begin{aligned}
&= \frac{1}{b_N a_{N+1} - b_{N+1} a_N} \left[\sum_{k=0}^{N-1} \left(\frac{b_N}{b_k a_N - b_N a_k} - \frac{b_{N+1}}{b_k a_{N+1} - b_{N+1} a_k} \right) \frac{b_k^{N-1}}{\prod_{\substack{l=0 \\ l \neq k}}^{N-1} (b_k a_l - b_l a_k)} \right. \\
&\quad \left. + \frac{b_N}{\prod_{l=0}^{N-1} (b_k a_l - b_l a_k)} - \frac{b_{N+1}}{\prod_{l=0}^{N-1} (b_k a_l - b_l a_k)} \right] \\
&= \frac{1}{b_N a_{N+1} - b_{N+1} a_N} \left[b_N \sum_{k=0}^N \frac{b_k^{N-1}}{\prod_{\substack{l=0 \\ l \neq k}}^N (b_k a_l - b_l a_k)} - b_{N+1} \sum_{\substack{k=0 \\ k \neq N}}^{N+1} \frac{b_k^{N-1}}{\prod_{\substack{l=0 \\ l \neq k, N}}^{N+1} (b_k a_l - b_l a_k)} \right] \\
&= 0.
\end{aligned} \tag{A.4}$$

Each of the sum in the last-but-one line vanishes by assuming the validity of Eq. (A.1). For the first sum this is obvious, for the second it should be realized that the sum follows from Eq. (A.1) by the substitutions $a_N \rightarrow a_{N+1}$ and $b_N \rightarrow b_{N+1}$.

In order to proof Eq. (3.4), we have to show

$$\prod_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} = \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{\hat{c}_{kn}^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^{N-1} (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}}. \tag{A.5}$$

Again we proceed by induction. For $N = 3$, there is nothing to show, as the l.h.s. and the r.h.s. are obviously equal. Now we assume the validity of Eq. (A.5) and proceed from N to $N + 1$ terms in the product on the l.h.s.,

$$\begin{aligned}
\prod_{\substack{k=0 \\ k \neq n, n+1}}^N \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} &= \frac{1}{-x_n \hat{c}_{Nn} + \hat{d}_{Nn}} \prod_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} \\
&= \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{\hat{c}_{kn}^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^{N-1} (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})} \frac{1}{(-x_n \hat{c}_{kn} + \hat{d}_{kn})(-x_n \hat{c}_{Nn} + \hat{d}_{Nn})}.
\end{aligned} \tag{A.6}$$

The last equation follows from assuming Eq. (A.5). Note that we have assumed $n \neq N, N - 1$ in the first manipulation, which is no loss of generality, since we can always achieve this by renumbering the propagators. Next we use the partial fraction

$$\frac{1}{(-x_n \hat{c}_{kn} + \hat{d}_{kn})(-x_n \hat{c}_{Nn} + \hat{d}_{Nn})} = \frac{1}{\hat{c}_{kn} \hat{d}_{Nn} - \hat{c}_{Nn} \hat{d}_{kn}} \left[\frac{\hat{c}_{kn}}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} - \frac{\hat{c}_{Nn}}{-x_n \hat{c}_{Nn} + \hat{d}_{Nn}} \right] \tag{A.7}$$

to obtain

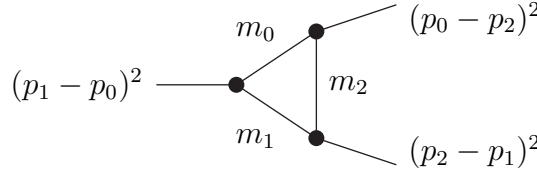
$$\prod_{\substack{k=0 \\ k \neq n, n+1}}^N \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} = \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{\hat{c}_{kn}^{N-2}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^N (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}}$$

$$\begin{aligned}
& - \frac{\hat{c}_{Nn}}{-x_n \hat{c}_{Nn} + \hat{d}_{Nn}} \sum_{\substack{k=0 \\ k \neq n, n+1}}^{N-1} \frac{\hat{c}_{kn}^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^N (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})} \\
& = \sum_{\substack{k=0 \\ k \neq n, n+1}}^N \frac{\hat{c}_{kn}^{N-2}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^N (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})} \frac{1}{-x_n \hat{c}_{kn} + \hat{d}_{kn}} \\
& - \frac{\hat{c}_{Nn}}{-x_n \hat{c}_{Nn} + \hat{d}_{Nn}} \sum_{\substack{k=0 \\ k \neq n, n+1}}^N \frac{\hat{c}_{kn}^{N-3}}{\prod_{\substack{l=0 \\ l \neq k, n, n+1}}^N (\hat{c}_{kn} \hat{d}_{ln} - \hat{c}_{ln} \hat{d}_{kn})}. \quad (\text{A.8})
\end{aligned}$$

The last equality follows from the fact that the terms with $k = N$ in the two sums cancel each other. Note that the last sum is identically zero owing to auxiliary relation (A.1) proven above. This completes the proof of Eqs. (A.5) and (3.4).

B Mass-singular scalar 3-point functions

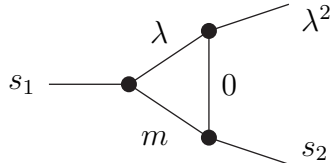
In this appendix we give a list of mass-singular scalar 3-point integrals that frequently appear in applications. Whenever the mass parameter λ appears, it is understood as infinitesimal. If not all singularities are cured by mass parameters, dimensional regularization is applied. Thus, in the following all formulas are valid up to order $\mathcal{O}(\lambda)$ or $\mathcal{O}(\epsilon)$. For convenience, a graphical notation is used,



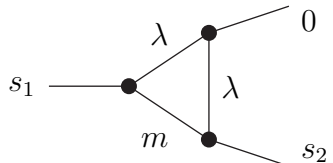
$$\begin{aligned}
& (p_1 - p_0)^2 \text{ --- } \bullet \text{ --- } \bullet \text{ --- } (p_0 - p_2)^2 \\
& \quad \quad \quad m_0 \quad \quad m_2 \\
& \quad \quad \quad \bullet \quad \quad \bullet \\
& \quad \quad \quad m_1 \quad \quad (p_2 - p_1)^2
\end{aligned}
\equiv C_0(p_0, p_1, p_2, m_0, m_1, m_2), \quad (\text{B.1})$$

and overlined variables are understood to receive an infinitesimally small imaginary part, $\bar{s} = s + i0$, etc. The function $\text{Li}_2(x) = -\int_0^x dt \ln(1-t)/t$ denotes the usual dilogarithm.

Collinear singularities show up in the following cases:

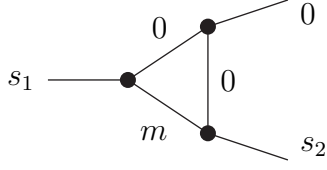


$$\begin{aligned}
& s_1 \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \lambda^2 \\
& \quad \quad \quad \lambda \quad \quad 0 \\
& \quad \quad \quad \bullet \quad \quad \bullet \\
& \quad \quad \quad m \quad \quad s_2
\end{aligned}
= \frac{1}{s_1 - s_2} \left\{ \ln \left(\frac{m^2 - \bar{s}_1}{\lambda^2} \right) \ln \left(\frac{m^2 - \bar{s}_1}{m^2} \right) \right. \\
& \quad - \ln \left(\frac{m^2 - \bar{s}_2}{\lambda^2} \right) \ln \left(\frac{m^2 - \bar{s}_2}{m^2} \right) - 2 \text{Li}_2 \left(\frac{\bar{s}_1 - \bar{s}_2}{m^2 - \bar{s}_2} \right) \\
& \quad \left. + \text{Li}_2 \left(\frac{\bar{s}_1}{m^2} \right) - \text{Li}_2 \left(\frac{\bar{s}_2}{m^2} \right) \right\}, \quad (\text{B.2})$$



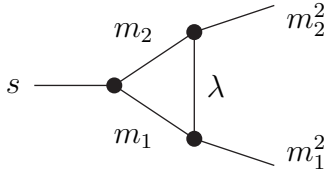
$$\begin{aligned}
& s_1 \text{ --- } \bullet \text{ --- } \bullet \text{ --- } 0 \\
& \quad \quad \quad \lambda \quad \quad \lambda \\
& \quad \quad \quad \bullet \quad \quad \bullet \\
& \quad \quad \quad m \quad \quad s_2
\end{aligned}
= \frac{1}{s_1 - s_2} \left\{ \ln^2 \left(\frac{m^2 - \bar{s}_1}{\lambda m} \right) - \ln^2 \left(\frac{m^2 - \bar{s}_2}{\lambda m} \right) \right\}$$

$$+ \text{Li}_2\left(\frac{\bar{s}_1}{m^2}\right) - \text{Li}_2\left(\frac{\bar{s}_2}{m^2}\right)\Big\}, \quad (\text{B.3})$$

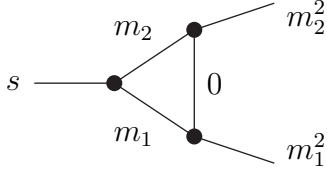


$$= \frac{1}{s_1 - s_2} \left\{ \frac{\Gamma(1 + \epsilon)}{\epsilon} \left(\frac{4\pi\mu^2}{m^2} \right)^\epsilon \ln\left(\frac{m^2 - \bar{s}_2}{m^2 - \bar{s}_1}\right) + \ln^2\left(\frac{m^2 - \bar{s}_1}{m^2}\right) - \ln^2\left(\frac{m^2 - \bar{s}_2}{m^2}\right) + \text{Li}_2\left(\frac{\bar{s}_1}{m^2}\right) - \text{Li}_2\left(\frac{\bar{s}_2}{m^2}\right) + \mathcal{O}(\epsilon) \right\}, \quad (\text{B.4})$$

There is only one situation for a pure soft singularity (given in two regularization schemes):



$$= \frac{x_s}{m_1 m_2 (1 - x_s^2)} \left\{ -\ln\left(\frac{\lambda^2}{m_1 m_2}\right) \ln(x_s) - \frac{1}{2} \ln^2(x_s) + 2 \ln(x_s) \ln(1 - x_s^2) + \frac{1}{2} \ln^2\left(\frac{m_1}{m_2}\right) - \frac{\pi^2}{6} + \text{Li}_2(x_s^2) + \text{Li}_2\left(1 - x_s \frac{m_1}{m_2}\right) + \text{Li}_2\left(1 - x_s \frac{m_2}{m_1}\right) \right\}, \quad (\text{B.5})$$

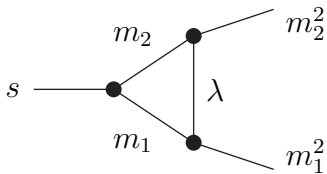


$$= \frac{x_s}{m_1 m_2 (1 - x_s^2)} \left\{ -\frac{\Gamma(1 + \epsilon)}{\epsilon} \left(\frac{4\pi\mu^2}{m_1 m_2} \right)^\epsilon \ln(x_s) - \frac{1}{2} \ln^2(x_s) + 2 \ln(x_s) \ln(1 - x_s^2) + \frac{1}{2} \ln^2\left(\frac{m_1}{m_2}\right) - \frac{\pi^2}{6} + \text{Li}_2(x_s^2) + \text{Li}_2\left(1 - x_s \frac{m_1}{m_2}\right) + \text{Li}_2\left(1 - x_s \frac{m_2}{m_1}\right) + \mathcal{O}(\epsilon) \right\}, \quad (\text{B.6})$$

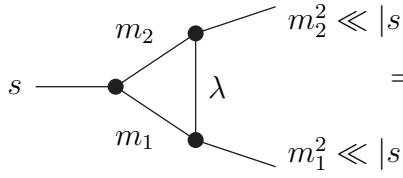
where Eq. (B.5) is taken over from Ref. [17] with

$$x_s = \frac{\sqrt{1 - 4m_1 m_2 / [\bar{s} - (m_1 - m_2)^2]} - 1}{\sqrt{1 - 4m_1 m_2 / [\bar{s} - (m_1 - m_2)^2]} + 1}. \quad (\text{B.7})$$

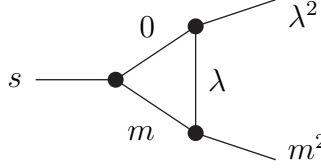
Finally, collinear and soft singularities can overlap:



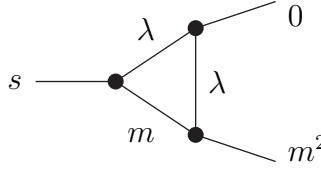
$$= \frac{1}{s - m_2^2} \left\{ \ln\left(\frac{m_1(m_2^2 - \bar{s})}{\lambda^2 m_2}\right) \ln\left(\frac{m_2^2 - \bar{s}}{m_1 m_2}\right) + \text{Li}_2\left(\frac{\bar{s}}{m_2^2}\right) \right\}, \quad (\text{B.8})$$



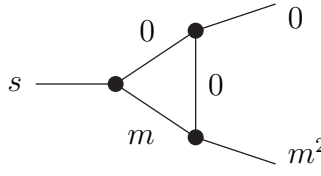
$$= \frac{1}{s} \left\{ \ln \left(\frac{\lambda^2}{-\bar{s}} \right) \ln \left(\frac{m_1 m_2}{-\bar{s}} \right) - \frac{1}{4} \ln^2 \left(\frac{m_1^2}{-\bar{s}} \right) - \frac{1}{4} \ln^2 \left(\frac{m_2^2}{-\bar{s}} \right) - \frac{\pi^2}{6} \right\}, \quad (\text{B.9})$$



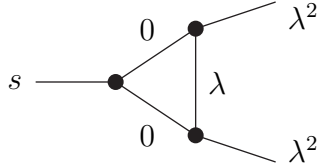
$$= \frac{1}{s - m^2} \left\{ \ln^2 \left(\frac{m^2 - \bar{s}}{\lambda m} \right) + \frac{5\pi^2}{12} + \text{Li}_2 \left(\frac{\bar{s}}{m^2} \right) \right\}, \quad (\text{B.10})$$



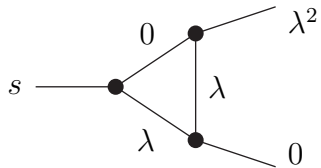
$$= \frac{1}{s - m^2} \left\{ \ln^2 \left(\frac{m^2 - \bar{s}}{\lambda m} \right) + \frac{\pi^2}{12} + \text{Li}_2 \left(\frac{\bar{s}}{m^2} \right) \right\}, \quad (\text{B.11})$$



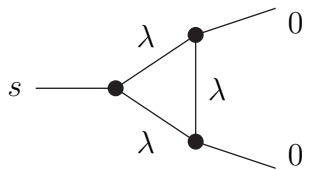
$$= \frac{1}{s - m^2} \left\{ \frac{\Gamma(1 + \epsilon)}{\epsilon^2} \left(\frac{4\pi\mu^2}{m^2 - \bar{s}} \right)^\epsilon - \frac{\Gamma(1 + \epsilon)}{2\epsilon^2} \left(\frac{4\pi\mu^2}{m^2} \right)^\epsilon - \text{Li}_2 \left(\frac{\bar{s}}{\bar{s} - m^2} \right) + \mathcal{O}(\epsilon) \right\}, \quad (\text{B.12})$$



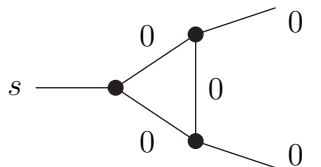
$$= \frac{1}{s} \left\{ \frac{1}{2} \ln^2 \left(\frac{\lambda^2}{-\bar{s}} \right) + \frac{2\pi^2}{3} \right\}, \quad (\text{B.13})$$



$$= \frac{1}{s} \left\{ \frac{1}{2} \ln^2 \left(\frac{\lambda^2}{-\bar{s}} \right) + \frac{\pi^2}{3} \right\}, \quad (\text{B.14})$$



$$= \frac{1}{2s} \ln^2 \left(\frac{\lambda^2}{-\bar{s}} \right), \quad (\text{B.15})$$



$$= \frac{1}{s} \left\{ \frac{\Gamma(1 + \epsilon)}{\epsilon^2} \left(\frac{4\pi\mu^2}{-\bar{s}} \right)^\epsilon - \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right\}. \quad (\text{B.16})$$

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